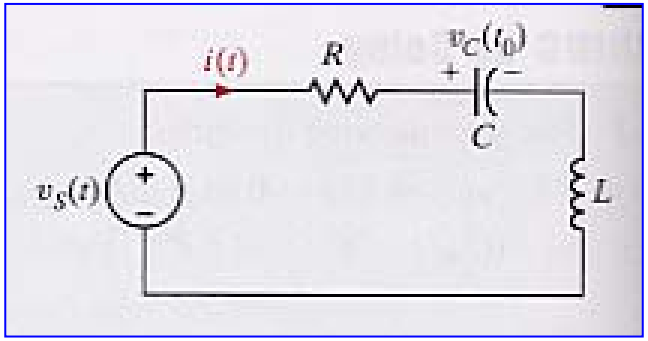
RLC Transient Response

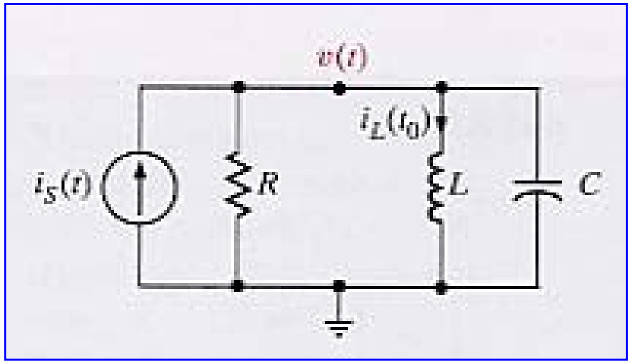
Second-order Circuits



Second-order series RLC circuit

$$Ri(t)+\frac{1}{C}\int_{t_0}^{t} i(x)dx+v_C(t_0)+L\frac{di(t)}{dt}=v_S(t)$$

$$\Rightarrow L\frac{d^2i(t)}{dt^2}+R\frac{di(t)}{dt}+\frac{i(t)}{C}=\frac{dv_S(t)}{dt}$$



Second-order parallel RLC circuit

$$\frac{v(t)}{R}+\frac{1}{L}\int_{t_o}^{t} v(x)dx+i_L(t_0)+C\frac{dv(t)}{dt}=i_S(t)$$

$$\Rightarrow C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{v(t)}{L}=\frac{di_S(t)}{dt}$$

General Solution

Differential equation으로 표현된 회로의 일반해

Homogeneous solution
(zero input response)

+

Particular solution
(zero initial state response)

입력을 가하지 않았을 때,
Initial value에 의한 response

충분히 긴 시간이 흘러 회로의
initial value가 없을 때,
특정 입력에 의한 response

General second-order differential equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = A$$

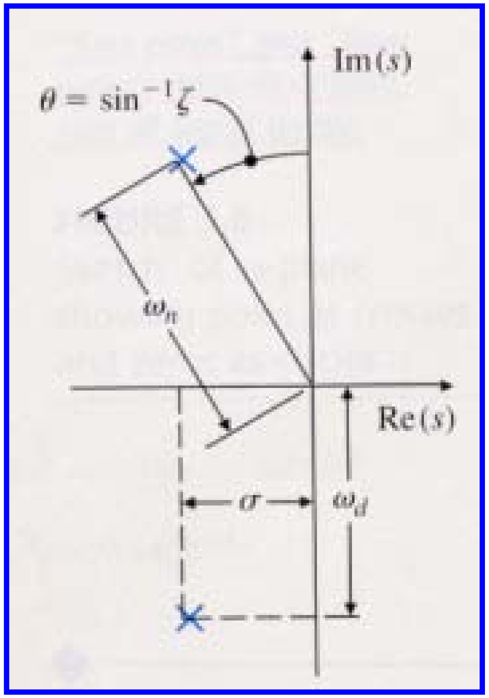
$$\begin{cases} x(t) = x_p(t) \quad : \quad particular \quad solution \quad for \quad \dfrac{d^2x(t)}{dt^2} + a_1\dfrac{dx(t)}{dt} + a_2x(t) = A \\ x(t) = x_c(t) \quad : \quad homogeneous \quad solution \quad for \quad \dfrac{d^2x(t)}{dt^2} + a_1\dfrac{dx(t)}{dt} + a_2x(t) = 0 \end{cases}$$

$$\Rightarrow General \quad solution \quad : \quad x(t) = x_p(t) + x_c(t)$$

Particular Solution

$$\frac{d^2x_p(t)}{dt^2}+a_1\frac{dx_p(t)}{dt}+a_2x_p(t)=A \;\Rightarrow\; x_p(t)=\frac{A}{a_2} \qquad \therefore x(t)=\frac{A}{a_2}+x_c(t)$$

Homogeneous Solution



<s-domain plot of roots>

$$\frac{d^2x_c(t)}{dt^2}+a_1\frac{dx_c(t)}{dt}+a_2x_c(t)=0 \;\Rightarrow\; \frac{d^2x_c(t)}{dt^2}+2\zeta\omega_0\frac{dx_c(t)}{dt}+\omega_0^{\,2}x_c(t)=0$$

$$Assume \quad x_c(t)=Ke^{st}$$

$$then \quad s^2Ke^{st}+2\zeta\omega_0 sKe^{st}+\omega_0^{\,2}Ke^{st}=0 \quad (a_1=2\zeta\omega_0,\, a_2=\omega_0^{\,2})$$

$$\Rightarrow \underline{s^2+2\zeta\omega_0 s+\omega_0^{\,2}=0} \;:\; charateristic \quad equation$$

$$roots \;:\; s=-\zeta\omega_0\pm j\omega_0\sqrt{1-\zeta^2}=-\sigma\pm j\omega_d \quad \begin{cases}\sigma=\zeta\omega_0\\ \omega_d=\omega_0\sqrt{1-\zeta^2}\end{cases}$$

- ζ : damping ratio(감쇠율) - 신호의 크기가 감쇠되는 정도를 나타냄
  - $\zeta > 1$: overdamping(과잉감쇠) – 정상상태에 도달하는데 긴 시간이 걸림
  - $\zeta = 1$: critical damping(임계감쇠) - 가장 빨리 정상 상태에 도달
  - $\zeta < 1$: underdamping(작은감쇠) – 진동하면서 exponential envelope에 따라 감쇠
- ω_0 : undamped natural frequency
  – ζ=0일 때, s=jω_0(purely imaginary) ⇒ 주기적으로 진동하는 sine wave

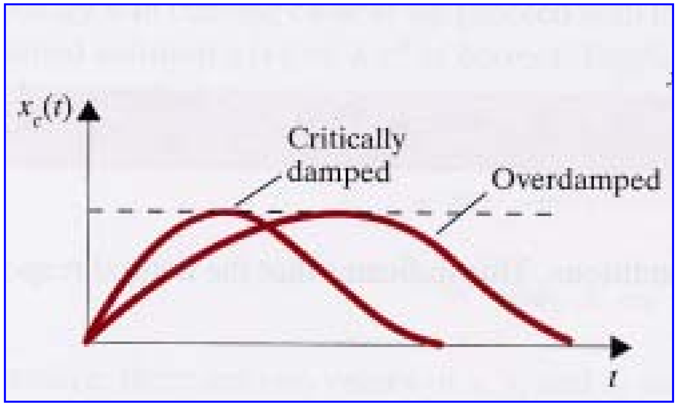
Homogeneous Solution - continue

$$two \quad roots: s_1 = -\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}, \quad s_2 = -\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}$$

$$then \quad x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \quad and \quad K_1, K_2 \quad is \quad obtained \quad from \quad initial \quad conditions$$

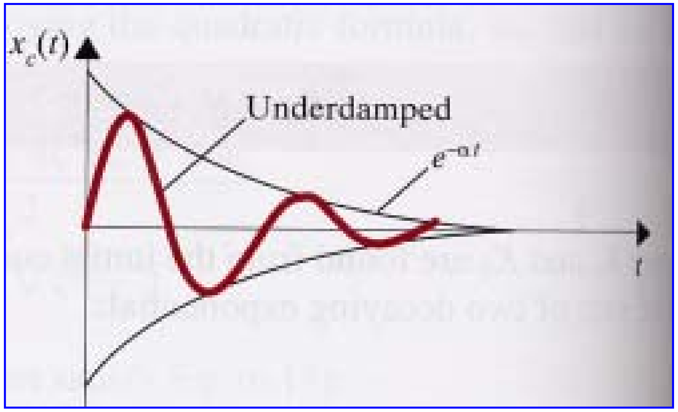
i) ζ > 1 : overdamping. s1, s2 are real and unequal.

$$x_c(t) = K_1 e^{-(\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1})t} + K_2 e^{-(\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1})t}$$



ii) ζ < 1 : underdamping. s1, s2 are complex conjugate

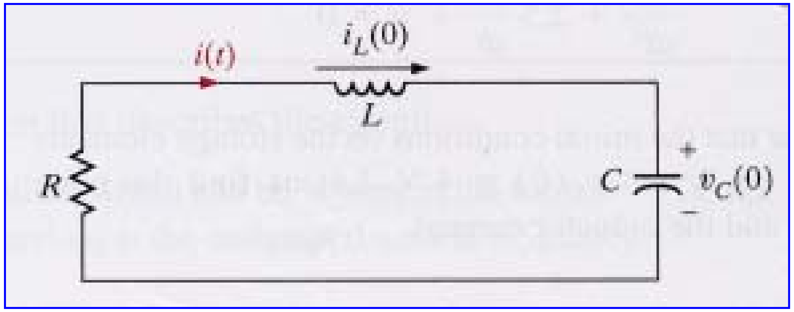
$$x_c(t) = e^{-\zeta\omega_0 t}(A_1 \cos\omega_0\sqrt{1-\zeta^2}t + A_2 \sin\omega_0\sqrt{1-\zeta^2}t)$$



iii) ζ = 1 : critical damping. s1 = s2 = $-\zeta\omega_0$

$$x_c(t) = B_1 e^{-\zeta\omega_0 t} + B_2 t e^{-\zeta\omega_0 t}$$

Example : Series RLC



R=6ohm, L=1H, C=0.04F, $i_L(0)$=4A, $v_C(0)$=-4V

i) second-order differential equation

$$\frac{d^2 i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{i(t)}{LC}=0 \longrightarrow \frac{d^2 i(t)}{dt^2}+6\frac{di(t)}{dt}+25i(t)=0$$

ii) Characteristic equation

$$s^2+6s+25=0 \;\Rightarrow\; s_1=-3+j4,\;\; s_2=-3-j4 \;:\; underdamping$$

iii) General solution

$$i(t)=K_1 e^{-3t}\cos 4t+K_2 e^{-3t}\sin 4t$$

iv) Applying initial conditions

$$i(0)=4=K_1$$

From KVL, $Ri(0)+L\frac{di(0)}{dt}+v_C(0)=0 \Rightarrow \frac{di(0)}{dt}=-20$

$$\frac{di(0)}{dt}=-3K_1+4K_2=-20 \Rightarrow K_2=-2$$

v) Final response

$$i(t)=4e^{-3t}\cos 4t-2e^{-3t}\sin 4t$$

From KVL, $v_C(t)=-4e^{-3t}\cos 4t+22e^{-3t}\sin 4t$